

The tidiness of elementary mathematics

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We all expect a certain *tidiness* to mathematics, and for many this provides a gratifying aesthetic pleasure. However, there are some wrinkles. One of these arises in the integral of x^k . Calculus provides us with the formula

$$\int x^k dx = \frac{x^{k+1}}{k+1}. \quad (1)$$

However, this equation is only correct for $k \neq -1$. If we try to take $k = -1$ the right hand side is meaningless because we have a zero on the denominator of the fraction, i.e. $\frac{1}{0}$. But in this case a separate argument gives the answer

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln(x). \quad (2)$$

Our expectation is that these two formulae should be reconciled. Indeed, if we let k approach -1 in the first we should end up with the second, but that fails to happen. For each x ,

$$\lim_{k \rightarrow -1} \frac{x^{k+1}}{k+1}$$

is undefined.

This is particularly problematic when we attempt to automate knowledge, that is when we seek to devise a *computer algebra system* which will symbolically manipulate expressions. If we want to integrate x^k with respect to x we certainly have two choices

1. Ignore the special case $k = -1$.
2. Ask the user if $k = -1$.

Both of these approaches are problematic. $k = -1$ is only one point out of an infinity of other values for k . However, in mathematics such exceptional cases occur rather more often than you might expect if you just stuck a pin in the number line, or even if you chose an integer k at random. Asking the user can also be problematic as they may not yet know the value of k , or the integral might occur deep within some other function and the user may have no idea what the question is asking! Automatically using the formula (1) might result in an error creeping into a much longer calculation. Mistakes like this are very hard to track down and can be serious.

What we would really like is a formula that works for all k . Here we are going to consider the comment in [2] that really, at a symbolic level, the answer to $\int x^k dx$ might better be written as

$$\int x^k dx = \frac{x^{k+1} - 1}{k+1} + c. \quad (3)$$

First consider the (easy) case $k \neq -1$. If we fix k and differentiate then we obtain

$$\frac{d}{dx} \frac{x^{k+1} - 1}{k+1} = x^k.$$

So $\frac{x^{k+1}-1}{k+1} = \frac{x^{k+1}}{k+1} - \frac{1}{k+1}$ is an anti-derivative of x^k . Really we have chosen a slightly different constant of integration.

Whereas we originally had the form $\frac{x^0}{0} = \frac{1}{0}$, the new formula gives $\frac{x^0-1}{0} = \frac{0}{0}$. To understand what is going on we need to look at this expression for k close to -1 and take a *limit*, i.e.

$$\lim_{k \rightarrow -1} \frac{x^{k+1} - 1}{k+1}.$$

If we can make sense of this limit, then we will define the value of the expression at $k = -1$ by the value of the limit, thus removing the exceptional case in a meaningful way. This sort of thing is a standard mathematician's move.

1 Euler's formula for $e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$

To consider the special case $k = -1$ we shall need to make use of a special formula discovered by Euler.

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n. \quad (4)$$

If we consider the Taylor series for e^x , on the left hand side we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

On the right, we can expand out the product using the Binomial Theorem

$$\left(1 + \frac{x}{n}\right)^n = \sum_{i=0}^n \frac{n(n-1) \cdots (n-i+1)}{i!} \left(\frac{x}{n}\right)^i = \sum_{i=0}^n \frac{x^i}{i!} \frac{n(n-1) \cdots (n-i+1)}{n^i}.$$

Consider the coefficients of x^i . This equals

$$\frac{1}{i!} \frac{n(n-1) \cdots (n-i+1)}{n^i} = \frac{1}{i!} \frac{n}{n} \frac{n-1}{n} \cdots \frac{n-i+1}{n}.$$

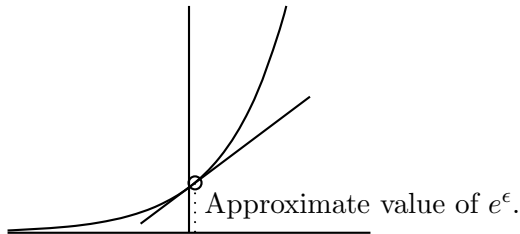
As $n \rightarrow \infty$ each term here containing n converges to 1 giving

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

or exactly the Taylor series for e^x .

You might think this argument has made somewhat cavalier use of limits. The following argument, adapted from [1, pg 272], is close to Euler's original. He was even more bold in his use of limits!

Euler unhesitatingly accepts the existence of both infinitely small and infinitely large numbers, and uses them to such effect that the modern reader's own hesitation must be tinged with envy. [1, pg 272]



Thinking about e^x near $x = 0$, and its gradient, we see that, for infinitely small ϵ

$$e^\epsilon = 1 + \epsilon$$

This is illustrated above. Let x be any given number then $N = x/\epsilon$ is infinitely large. So

$$e^x = e^{N\epsilon} = (e^\epsilon)^N = (1 + \epsilon)^N = \left(1 + \frac{x}{N}\right)^N.$$

Written in terms of limits, the infinitely large N would give us (4).

2 The awkward integral when $k = -1$

We shall make use of Euler's limit to evaluate the awkward integral when $k = -1$.

$$x = \left(1 + \frac{y}{n}\right)^n. \quad (5)$$

Rearranging this we have

$$x^{\frac{1}{n}} = 1 + \frac{y}{n}$$

and then

$$y = n \left(x^{\frac{1}{n}} - 1\right).$$

Let $m = \frac{1}{n}$ so

$$y = \frac{1}{m} (x^m - 1) = \frac{x^m - 1}{m}.$$

Now, set $m = k + 1$, to give

$$y = \frac{x^{k+1} - 1}{k + 1}$$

and we are coming somewhat closer to the right hand side of (3).

If $k \rightarrow -1$, then $m \rightarrow 0$ and $n = \frac{1}{m} \rightarrow \infty$. But if $n \rightarrow \infty$ then we may apply (4) to (5) and conclude that $x = e^y$. Taking the inverse gives $y = \ln(x)$. Hence

$$\lim_{k \rightarrow -1} \frac{x^{k+1} - 1}{k + 1} = \ln(x)$$

and so (3) is indeed valid for all k , including the special case $k = -1$, when we define the value of (3) in this case by the limit $\lim_{k \rightarrow -1}$. This limit happens to give the right answer to the integral for $k = -1$. In this sense (3) is somewhat better than (1).

3 The mathematical moral

The key to all of this is to choose the constant of integration in (1) so that the resulting formula also holds in the awkward limiting case, $k = -1$. Like many tricks in mathematics it is a technique which occasionally has useful applications elsewhere.

There is a deep consistency to elementary mathematics, sometimes we just have to search to find it.

References

- [1] C. H. Edwards. *The Historical Development of the Calculus*. Springer-Verlag, 1979.
- [2] D. R. Stoutemyer. Crimes and misdemeanors in the computer algebra trade. *Notices of the American Mathematical Society*, 38(7):778–785, September 1991.